# TASI Lecture 2: Extensions and Applications of TFT

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# 1. Outline

## 2. Classifying Spaces & Group Cohomology

We return to finite group gauge theory and give another perspective.

For any topological group $G$ there is a topological space (only defined up to homotopy equivalence) denoted $BG$ with the property that there is a 1-1 correspondence:

$$\left\{ \begin{array}{l} \text{homotopy classes} \\ \text{of maps} \\ \quad f: M \to BG \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of principal } G\text{-bundles} \\ P \to M \end{array} \right\}$$

One way to construct $BG$ is to find a space $EG$ which is

(a) contractible

(b) admits a free $G$-action.

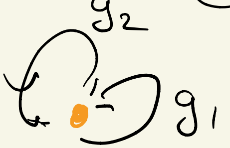
Then $BG = EG/G$.

For example, for $G = \mathbb{Z}$ we can take $EG = \mathbb{R}$ with $\mathbb{Z}$ acting by translations, so $BG$ is any topological space h.e. to $S^1$. But this example is a bit misleading. Consider $G = \mathbb{Z}_2$. The simplest model for $EG$ is the unit sphere in an $\infty$-dim'l Hilbert space and $B\mathbb{Z}_2 \sim \mathbb{RP}^\infty$.

A systematic way to proceed begins by identifying a group with a category

$$C_0 = \{\bullet\} \qquad C_1 = \mathrm{Hom}(\bullet, \bullet) = G$$

The composition of arrows in $C_1$ is defined by group multiplication




$$g_1 \circ g_2 = g_1 g_2$$

Now assume $G$ is a finite group.

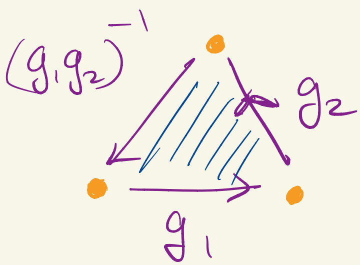
We construct a CW complex

0 – skeleton $= *$

1 – skeleton $= G$

Next the aim is attach disks of higher dimension so that all the higher homotopy groups vanish and $\pi_1(BG) \cong G$. but $\pi_j(BG) = 0 \quad j > 1$.

We view $C_2$ as defining triangles which we wish to fill in.




(arrow $\bullet \xrightarrow{g} \bullet$ with reversed orientation is $\bullet \xrightarrow[g^{-1}]{} \bullet$)

In equations, take the $n$-simplex

$$\Delta_n = \{ (t_0, \dots, t_n) \mid t_i \geq 0 \;\&\; \sum_0^n t_i = 1 \}$$

and consider the $\infty$ space

$$\coprod_{n=0}^{\infty} \Delta_n \times G^n$$

We introduce maps

$$\begin{tikzcd} \bullet & G \arrow[l] & G\times G \arrow[l, shift left=2, "d^0"'] \arrow[l, "d^1"] \arrow[l, shift right=2, "d^2"] & G\times G\times G \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \end{tikzcd}$$

$$d^0(g_1, \dots, g_n) = (g_2, \dots g_n)$$
$$d^1(g_1, \dots g_n) = (g_1 g_2, g_3, \dots, g_n)$$
$$\vdots$$
$$d^{n-1}(g_1, \dots, g_n) = (g_1, \dots, g_{n-1} g_n)$$
$$d^n(g_1 \dots g_n) = (g_1, \dots g_{n-1})$$

and opposite face maps $d_i : \Delta_{n+1} \longrightarrow \Delta_n$ which put $t_i = 0$.

$$BG \sim \left(\coprod_{n=0}^{\infty} \Delta_n \times G^n\right) \Big/ \left(d_i(\vec{t}), \vec{g}\right) \sim \left(\vec{t}, d^i(\vec{g})\right)$$

Given the existence of $BG$ we can view finite $G$ gauge theory as a "nonlinear $\sigma$-model up to homotopy"

$$Z(M_n) = \sum_{\pi_0(\mathrm{Map}(M_n \to BG))} \frac{1}{|\mathrm{Aut}\,\phi|}$$

$$Z(N_{n-1}) = \mathrm{Functions}\left(\pi_0\left(\mathrm{Map}(N_{n-1} \to BG)\right)\right)$$

# 3. Group Cohomology

The cohomology of BG defines the group cohomology. Concretely:

$$C^n(G, A) = \{ \phi : G^n \longrightarrow A \}$$

Abelian

$$\delta : C^n(G,A) \longrightarrow C^{n+1}(G,A)$$

$$\begin{aligned}(\delta\phi)(g_1, \dots, g_{n+1}) = \ & \phi(g_2, \dots g_{n+1}) \\ & - \phi(g_1 g_2, g_3, \dots, g_{n+1}) \\ & + \phi(g_1, g_2 g_3, \dots, g_{n+1}) \\ & \pm \dots + (-1)^{n+1} \phi(g_1, \dots, g_n)\end{aligned}$$

Exercise: (a) Check $\delta^2 = 0$

(b.) write out $\delta\phi = 0$ for the first few cases

Def: $H^n(G, A) = \ker\delta / \operatorname{im}\delta$

# 4. Digression: Projectivity In Quantum Mechanics

One of the most important group cohomologies in physics is $H^2(G, U(1))$, which classifies iso. classes of central extensions.

One of the foundations of quantum mechanics is the Born rule:

1. Physical states $\rho$ are traceclass positive op's on a $\mathbb{C}$-Hilbert space $\mathcal{H}$ with trace $\mathrm{tr}(\rho) = 1$
2. Physical observables $\mathcal{O}$ are self-adjoint operators on $\mathcal{H}$.

The Born rule is a pairing of states + observables to give a probability distribution on $\mathbb{R}$: For a measurable set

$E \subset \mathbb{R}$
$$P_{\rho,\mathcal{O}}(E) = \mathrm{Tr}_{\mathcal{H}}\left(P_{\mathcal{O}}(E)\,\rho\right)$$

where $P_{\mathcal{O}}(E)$ is the projection-valued measure associated to $\mathcal{O}$ by the spectral theorem. If $\mathcal{O}$ has a discrete spectrum of eigenvalues $\lambda_i$

$$P_{\mathcal{O}}(E) = \sum_{\lambda_i \in E} P(\lambda_i)$$

$P(\lambda_i)$ = Projector onto eigenspace for $\lambda_i$

An <u>automorphism</u> of a quantum system is a bijective map of states + observables preserving the Born rule.

One can show it is determined by a bijective correspondence on <u>pure states</u> preserving the overlap function

$$o(P_1, P_2) = \mathrm{Tr}(P_1 P_2) = \frac{|\langle\psi_1|\psi_2\rangle|^2}{\|\psi_1\|^2\|\psi_2\|^2}$$

for pure states $P_i = \dfrac{|\psi_i\rangle\langle\psi_i|}{\|\psi_i\|^2}$

The pure states form a <u>projective</u> Hilbert space and the overlap function is related to the Fubini-Study metric on projective space:

$$\mathcal{O}(P_1, P_2) = \cos^2 \frac{d_{FS}(P_1, P_2)}{2}$$

So the automorphism group of a quantum system is the group of isometries of complex projective space.

Wigner's theorem relates this to (anti-) linear operators on $\mathcal{H}$.

Let $Aut(\mathcal{H})$ = group of unitary and anti-unitary op's on $\mathcal{H}$.

$$\pi : Aut(\mathcal{H}) \longrightarrow Aut(QM)$$

$$g \longmapsto \pi(g) : P \to g P g^{-1}$$

Wigner's theorem asserts that $\pi$ is surjective and the kernel is the group $U(1)$ acting as scalars on $\mathcal{H}$.

$$\begin{tikzcd} 1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}(\mathcal{H}) \arrow[r, "\pi"] & \mathrm{Aut}(QM) \arrow[r] & 1 \\ & & & G \arrow[u, "\varphi"'] & \end{tikzcd}$$

Now, if we have dynamics only a subgroup of $\mathrm{Aut}(QM)$ will commute with the flows on $\mathcal{S}, \mathcal{O}$.

By Wigner's theorem $\forall g \in G$ we can pick a $U(g) \in \mathrm{Aut}(\mathcal{H})$ s.t.

$$\pi\big(U(g_1)U(g_2)\big) = \pi\big(U(g_1 g_2)\big)$$

But this only allows us to conclude that

$$U(g_1)U(g_2) = c(g_1, g_2)\, U(g_1 g_2)$$

for some function $c : G \times G \to U(1)$

Now assume (for simplicity) that the $U(g)$ are $\mathbb{C}$-linear for $g \in G$.

Then (exercise!) $c \in Z^2(G, U(1))$

The pull back group

$\tilde{G} = U(1) \times G$ with group law

$$(z_1, g_1)(z_2, g_2) := (z_1 z_2 c(g_1, g_2), g_1 g_2)$$

is linearly represented on $\mathcal{H}$

$$T((z, g)) = z\, U(g).$$

A good example is a spin $1/2$ Qbit where the $SO(3)$ isometry of $\mathbb{CP}^1$ is represented on the Hilbertspace $\mathbb{C}^2$ by the central extension

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1.$$

# 5. Dijkgraaf-Witten Theory

DW gave a "lattice gauge theory model" of topological finite $G$ gauge theory, and an important generalization thereof based on group cohomology.

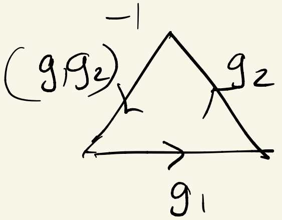
Let us just describe it for 2-dimensions, and we just explain how to compute the partition function. The construction generalizes to "fully extended" (see below) $n$-dim'l theories.

Let $M_2$ be an oriented compact surface

Choose a triangulation on $M_2$

Require that the gauge field be flat

So the plaquette Boltzmann weight is only determined by two group elements

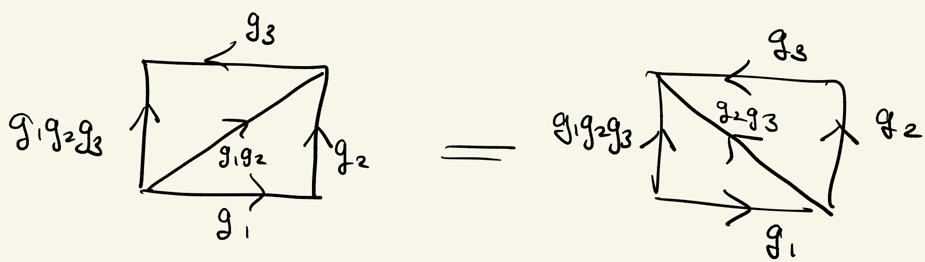



has weight $W(g_1, g_2) \in \mathbb{C}^*$

(some choices must be made for this to be unambiguous).

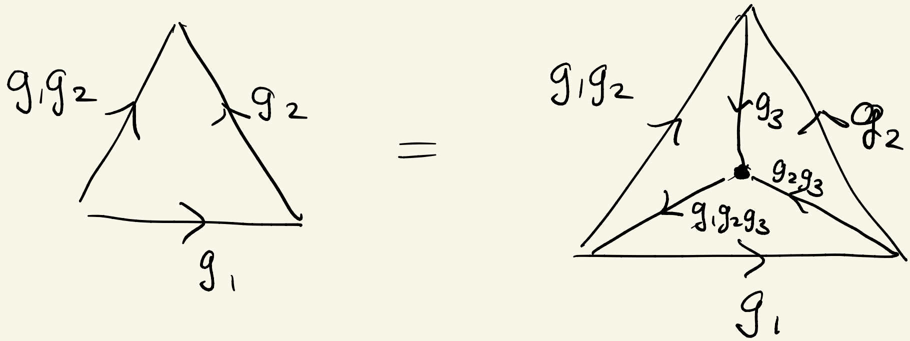
$$F(M_2) = \sum_{\text{maps}: \text{Edges} \to G} \prod_{\Delta} W(g_1, g_2)$$

Now demand invariance under triangulation:




$\Rightarrow W(g_1, g_2)$ is a group cocycle

then one checks, using the cocycle identity, that we have invariance under refinement:




Which can be viewed as a kind of "renormalization group fixed point" condition.

<u>Fact:</u> All triangulations can be obtained by these two moves

So $F(M_2)$ is independent of triangulation

Moreover $F(M_2)$ only depends on the group cohomology class determined by $W(g_1, g_2)$.

In $n$-dimensions we use a simplicial decomposition and the Boltzmann weights are an $n$-cocycle on $G$ valued in $\mathbb{C}^*$.

In particular, in 3 dimensions the theory is determined by an element of $H^3(G, U(1))$.
For a finite group one has:

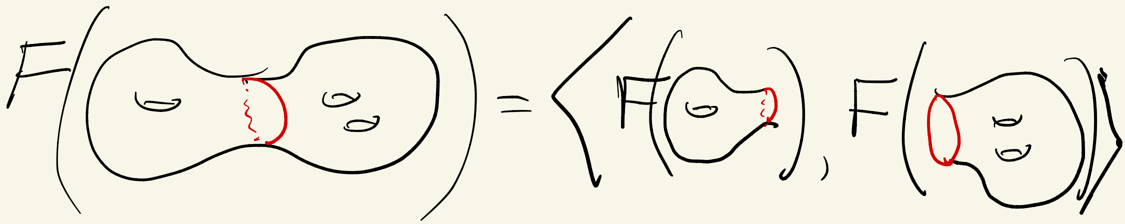
$$H^3(G, U(1)) \cong H^4(G, \mathbb{Z})$$

So the $n=3$ case is just Chern-Simons-Witten theory for a finite group $G$.
Quite generally, for a compact group $G$ the 3d CSW is completely determined by a "level"

$$k \in H^4(G, \mathbb{Z}).$$

# 6. Higher Categories

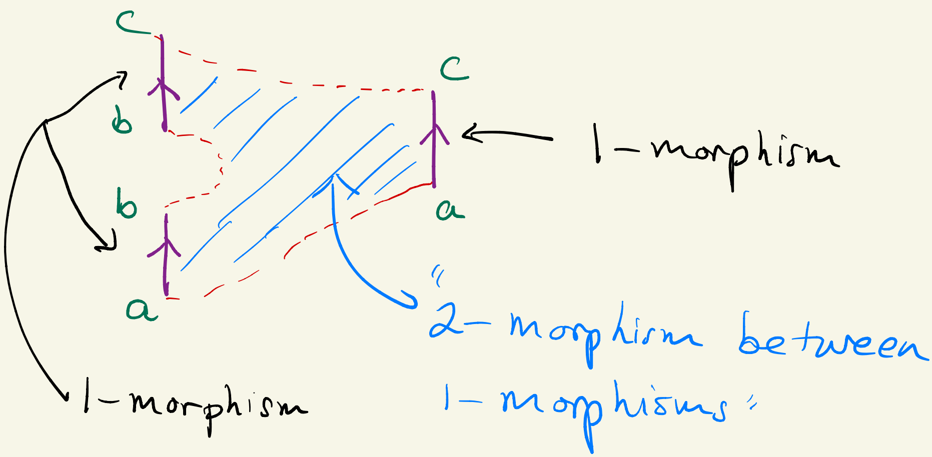
Now we want to start describing *extended* TFT. The idea is to take locality to its logical limit. We have the gluing formula relating partition functions to pairings of vectors in state spaces



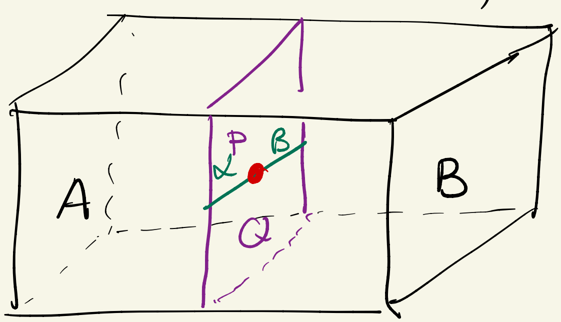
The question then naturally arises whether the space of states associated to a compact $N_{n-1}$ w/out boundary can likewise be assembled from pieces

$$F\left(N_{n-1} = N^{0}_{n-1} \underset{\Sigma_{n-2}}{\cup} N'\right) = \, ?$$

A second way of motivating the higher categories comes from considering 2d open/closed theory:




A third way of motivating these ideas comes from thinking of defects within defects (Kapustin ICM 2010)



<u>A fourth way</u> comes from thinking about the proper categorical interpretation of the fundamental group:

Let $X$ be a topological space. We form a <u>category</u> whose objects are the points of $X$. The morphisms $x_1 \to x_2$ are paths in $X$ $\gamma : x_1 \rightsquigarrow x_2$ considered up to homotopy with fixed endpoints $\mathrm{Hom}(x_1, x_2) = P(x_1, x_2) / \text{homotopy}$

Then $\pi_1(X, x_0) = \mathrm{Hom}(x_0, x_0) =: \mathrm{Aut}(x_0)$.

This (important) category is called the fundamental groupoid $\pi_{\leq 1}(X)$.

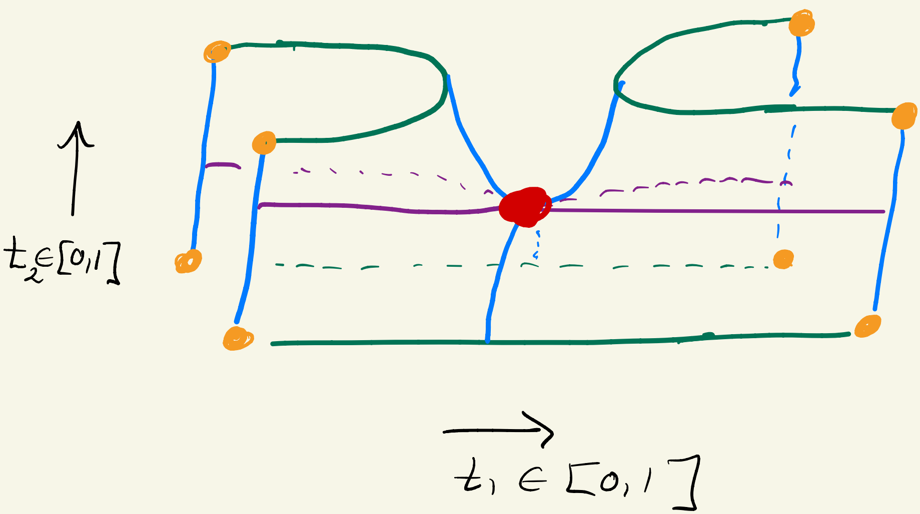
But we could make a more elaborate object if we decline to consider paths only up to homotopy.

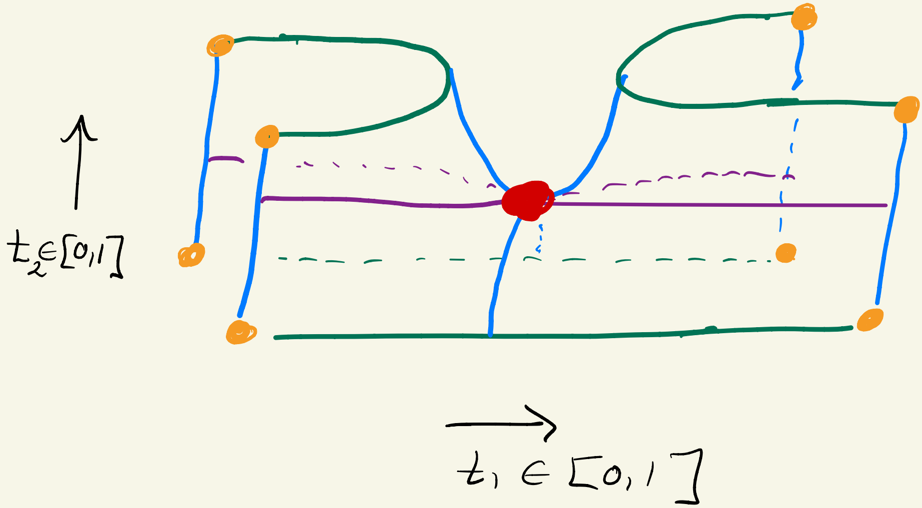
We could consider $\pi_{\leq 2}(X)$

where 1-morphisms between objects $x_1 \to x_2$ $P(x_1, x_2)$ and "2-morphisms" are homotopies of paths.

And so on, up to $\pi_\infty(X)$.

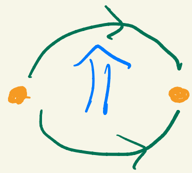
<u>A fifth way</u> originates from Morse theory. Let us revisit the topology change induced by a saddle:



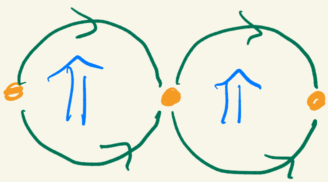


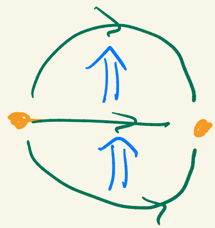

The zero objects are the orange points.
At time $t_2 = 0$ we have a bordism from two points to two points colored in green. It is the evolution along $t_1$. Recall that the bordism is a 1-morphism in the category $\mathrm{Bord}_{\langle 0,1\rangle}$.
At time $t_2 = 1$ we have a bordism between the same pair of objects we had at $t_2 = 0$, but the green bordism is a different 1-morphism in $\mathrm{Bord}_{\langle 0,1\rangle}$.
The saddle is a 2-morphism between these two 1-morphisms.

In general if we think of categories in terms of directed graphs when we add 2-morphisms we introduce a new kind of arrow:



These can be composed in several different ways





and there are <u>many</u> <u>many</u> technicalities (e.g. rigidity etc.) and many axioms

But (see references) one can extend the idea of a category to an n-category.

In an $n$-category the Hom-spaces between objects (a.k.a. "0-morphisms") are $(n-1)$-categories.

Remark: There is a refinement of this notion to a $(p,q)$ category:

All $k$-morphisms with $k>p$ are the identity

All $k$-morphisms with $k>q$ are invertible

In this notation a $(0,0)$ category is a set. A $(1,1)$ category is a category in the normal sense.

One commonly encounters the term "$(\infty,n)$-category" so it is an $\infty$-category where all $k$-morphisms $k>n$ are invertible.

$\pi_\infty(X)$ is an $(\infty,0)$-category.

For careful definitions see works of Barwick, Bergner, Schommer-Pries, Rezk, and references in the surveys by Freed, Safronov, and Teleman.

Example 1 A good example of a 2-category is the category $ALG(VECT)$:

0-morphisms are algebras $A$

1-morphisms are $A_1 - A_2$ bimodules

2-morphisms are bimodule maps.

Example 2: A second useful example of a 2-category is CAT

0-morphisms are (small) $\mathbb{C}$-linear categories

1-morphisms are functors

2-morphisms are $\natural$ trans between functors

Example 3: By extending the discussion of the saddle above we can define an $n$-category $Bord_n$ by taking the $k$-morphisms to be bordisms with $k$-times.

This can even be generalized to an $(\infty, n)$ category by taking $(n+1)$-morphisms to be diffeomorphisms of bordisms preserving all the initial and final $k$-bordisms.

The $(n+2)$-morphisms are isotopies of diffeos etc.

Finally the bordisms can be endowed with background fields to define $\mathrm{Bord}_n(\mathcal{F})$.

# Monoidal Structure:

Finally, the notion of monoidal (tensor) category can be extended to n-categories. This requires hard work.

For the 2-category ALG(VECT) the monoidal structure is the usual $\otimes$ product of algebras, bimodules, and linear maps.

For $\mathrm{Bord}_n(\mathcal{F})$ the monoidal structure is disjoint union.

# 7. Extended TFT

Let $\mathcal{C}$ be a (symmetric) monoidal $n$-category

Then an extended TFT is a monoidal functor of $n$-categories

$$F : \mathrm{Bord}_{\leq n}(\mathcal{F}) \longrightarrow \mathcal{C}$$

Monoidal $n$-categories have a distinguished 0-morphism $1_{\mathcal{C}}$, the unit under $\otimes$ and one defines the looped category

$$\Omega\mathcal{C} := \mathrm{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}})$$

which is a monoidal $(n-1)$-category.

In our discussion here we will assume that

$$\Omega^{n-1}\mathcal{C} = \mathrm{VECT} \implies \Omega^{n}\mathcal{C} = \mathbb{C}.$$

With this notation if $M_k$ is a compact $k$-manifold without boundary then

$$F(M_k) \in \mathrm{Obj}(\Omega^k \mathcal{C})$$

An important point is that there can be different $n$-categories $\mathcal{C}$ with $\Omega^{n-1}\mathcal{C} = \mathrm{VECT}$. So the <u>choice</u> of codomain is an important part of specifying an extended TFT.

<u>Example:</u> For finite gauge theory in $n=2$ dimensions

$$\mathcal{C} = \mathrm{ALG}(\mathrm{VECT}) \qquad F(pt) = \mathbb{C}[G]$$

$$\mathcal{C} = \mathcal{C}AT \qquad F(pt) = \mathrm{VECT}(\pi_{\leq 1}(BG))$$

$$= \mathrm{Rep}(G) =$$

category of reps of $G$.

Remark: The "cobordism hypothesis" is an idea going back to Baez & Dolan. It states, very roughly, that a fully extended TFT is "completely determined by its value on a point"

Recall that $F(pt) \in \mathrm{Obj}(\mathcal{C})$ is an object in an $n$-category.

A good example is the case $n=1$ where $F(pt) = V$, a vector space w/ nondegenerate bilinear form defines the theory.

A precise version was proved by Jacob Lurie. We just give a very rough idea:

1.) To every $(\infty,0)$ category $C$ we can assign a topological space $sp(C)$ so that there is an equivalence of $C$ with $\pi_\infty(sp(C))$.

2.) Given a fixed $(\infty,n)$ category, $\mathcal{C}$, the codomain, one can define an $(\infty,0)$ category of topological field theories $Hom(Bord_n, \mathcal{C})$ and therefore there is a corresponding

"Space of Theories" $X$

$$X = Sp(\mathrm{Hom}(\mathrm{Bord}_n, \mathcal{C}))$$

Note: This realizes, in this setting, an old dream of physicists of defining a "space of QFT's."

Given an $(\infty, n)$ category $\mathcal{C}$ there is:

1.) A subcategory $\mathcal{C}^{fd}$ of "finite dimensional" $k$-morphisms: These satisfy the analog of the S-diagram argument from lecture 1.

2.) An $(\infty, 0)$ category $(\mathcal{C}^{fd})^{\sim}$

obtained by deleting all noninvertible $k$-morphisms.

Finally, we must endow $\mathrm{Bord}_n$ with the "field" of a framing. This means $k$-bordisms $W$ have a trivialization

$$TW \oplus (W \times \mathbb{R}^{n-k}) \simeq W \times \mathbb{R}^n$$

The cobordism hypothesis states there is a homotopy equivalence of topological spaces

$$\mathrm{Sp}\left(\mathrm{Hom}(\mathrm{Bord}_n(\mathrm{fr})), \mathcal{C}\right) \simeq \mathrm{Sp}\left(\widetilde{\mathcal{C}^{fd}}\right)$$

given by $F \longmapsto F(pt)$

# 8. Finite Homotopy Theories

## 8a: $\pi$-finite spaces.

When we discussed finite group gauge theory we introduced the space $BG$. It has the property that

$$\pi_q(BG) \cong \begin{cases} \{1\} & q>1 \\ G & q=1 \end{cases}$$

There is a generalization available when $G = A$ is an Abelian group.

For every Abelian group $A$ and integer $n > 1$ we can define an "Eilenberg-MacLane space" $K(A, n)$ (up to h.e.) by

$$\pi_q(K(A,n)) \cong \begin{cases} \{1\} & n \neq q \\ A & n = q \end{cases}$$

e.g. $K(\mathbb{Z}, 1) = S^1$, but this is atypical:

$$K(\mathbb{Z}, 2) \neq S^2$$

after all $\pi_3(S^2) \cong \mathbb{Z}$ !

$$\pi_{j \geq 4}(S^2) = \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_{12}, \mathbb{Z}_2, \ldots$$

So to construct $K(\mathbb{Z},2)$ we would need to attach higher and higher disks to kill the higher homotopy groups.

A better way to think about it: Consider the set of pure states in an n-dim'l Hilbert space

$$\mathbb{CP}^n = S^{2n+1}/U(1)$$

LES of homotopy groups $\Rightarrow$

$$\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$$

$$\pi_j(\mathbb{CP}^n) = 0 \quad j = 3, \dots, 2n+1$$

"So, take the $n \to \infty$ limit"

We can identify $K(\mathbb{Z},2)$ with the space of pure states in $\infty$-dim'l Hilbert space.

$K(A,n)$ will typically have some kind of infinite-dimensional model.

We can now start thinking about $K(A,n)$ bundles over topological spaces $X$. These are classified by homotopy classes

$$X \longrightarrow K(A,n+1)$$

**Def:** A $\pi$-finite space $\mathcal{X}$ is a topological space with a finite set of connected components, each of which has a finite set of nonzero homotopy groups $\pi_j(\mathcal{X}_\alpha)$ each of which is a finite group.

When $\mathcal{X}$ is connected it has a "Postnikov decomposition" as an iterated fibration of Eil-Mach spaces: (See Bott + Tu pp. 250-251)

$$\begin{tikzcd}
K(\pi_3, q_3) \arrow[r] & \mathcal{X}^{(3)} \arrow[d] & \\
K(\pi_2, q_2) \arrow[r] & \mathcal{X}^{(2)} \arrow[r] \arrow[d] & BK(\pi_3, q_3) = K(\pi_3, q_3+1) \\
& \mathcal{X}^{(1)} = K(\pi_1, q_1) \arrow[r] & BK(\pi_2, q_2) = K(\pi_2, q_2+1)
\end{tikzcd}$$

A 2-stage Post. decomp of the form

$$\begin{tikzcd} K(A,2) \arrow[r] & \mathcal{X} \arrow[d] & \\ & K(G,1) \arrow[r] & K(A,3) \end{tikzcd}$$

is called a "2-group" and plays an important role in Dumitrescu's lectures.

In general, $\pi$-finite spaces are also referred to as "higher groups".

# 8b: The TFT's $\sigma^{(m)}_{\mathcal{X},\mathcal{C}}$

Given a $\pi$-finite space $\mathcal{X}$ and a symmetric monoidal $m$-category $\mathcal{C}$ one can construct an $m$-dim'l extended TFT denoted $\sigma^{(m)}_{\mathcal{X},\mathcal{C}}$.

For the case $\mathcal{C}$ is a "Morita $m$-category" (constructed from algebra objects) a fairly complete description is in Freed–Hopkins–Lurie–Teleman

Assuming $\Omega^{m-1}\mathcal{C} = \mathrm{VECT}$
a concrete description of the "top two levels" is the following

Notation: For any manifold $M$
let $\mathcal{X}^M := \text{Cont. Map}(M \to \mathcal{X})$

Then we define the state-spaces:

$$\sigma^{(m)}_{\mathcal{X}}(N_{m-1}) := \mathrm{Fun}\left(\pi_0\left(\mathcal{X}^{N_{m-1}}\right)\right)$$

To see this is reasonable consider the quantization of the $m$-dimensional scalar field with action $\sim \int_{M_m} (\partial\phi)^2$

The theory depends on a metric and the Hilbert space of the theory on a compact manifold without bdry $N_{m-1}$ should be something like $L^2(\mathcal{X}^{N_{m-1}})$. States would be derived from wavefunctionals

$$\Psi[\phi(x)] \qquad \phi \in \mathcal{X}^{N_{m-1}}$$

Here in the TFT setting we are only working up to homotopy

Examples:

1.) $\mathcal{X} = K(A, q)$

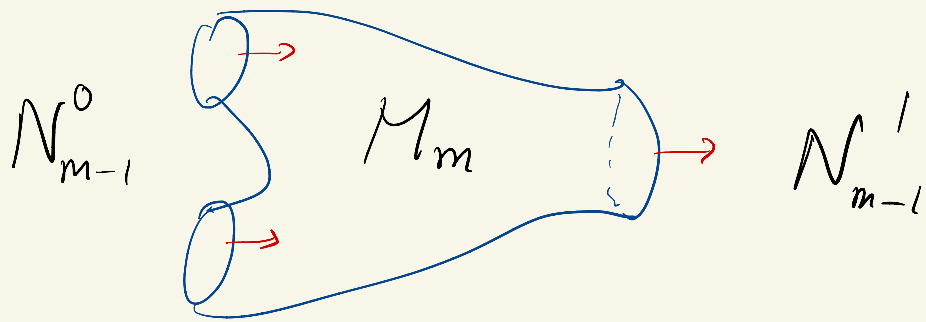
$$\pi_0\left(\mathcal{X}^{N_{m-1}}\right) \cong H^q(N_{m-1}, A)$$

So the "space of states of $\sigma^{(m)}_{\mathcal{X}}$" on the spatial manifold $N_{m-1}$ is the vector space of functions from the finite Abelian group $H^q(N_{m-1}, A)$ to the complex numbers.

2.) $\mathcal{X} = K(G, 1) = BG$

$$\pi_0\left(\mathcal{X}^{N_{m-1}}\right) = \left\{\begin{array}{l}\text{isom. classes} \\ \text{of principal} \\ G\text{-bundles over} \\ \text{spatial } N_{m-1}\end{array}\right.$$

Now to define amplitudes associated to a bordism:




we form a "correspondence"

$$\begin{tikzcd} & \mathcal{X}^{M_m} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \mathcal{X}^{N^0_{m-1}} & & \mathcal{X}^{N^1_{m-1}} \end{tikzcd}$$

$p_0, p_1$ are given by restricting the field $\phi \in \mathcal{X}^{M_m}$ to the in- and out- boundaries

The idea is that the linear map

$$F(M_m) : F(N^0_{m-1}) \to F(N^1_{m-1})$$

is given by pullback + pushforward

$$F(M_m) = (p_{1,*}) \circ p_0^*$$

While $p_0^*$ is straightforward $p_{1,*}$ is not; It uses the "homotopy cardinality"

$$p_{1,*}(\psi)(h) = \sum_{[\phi] \in \pi_0(p_1^{-1}(h))} \left( \prod_{i=1}^{\infty} \left| \pi_i(p_1^{-1}(h), \phi) \right|^{(-1)^i} \right) \psi(h)$$

Using properties of homotopy fiber products one can check the crucial gluing properties.

<u>Remark</u>: Taking $N^0 = N^1 = \phi$ gives the partition function on a compact $m$-manifold w/out bdry as a corollary:

$$F(M_m) = \sum_{[\phi] \in \pi_0(\mathcal{X}^{M_m})} \left( \prod_{i=1}^{\infty} \left| \pi_i(\mathcal{X}^{M_m}, \phi) \right|^{(-1)^i} \right)$$

<u>Remark</u>: Correspondences and homotopy fiber products play a crucial role in this subject.

In general, a <u>correspondence</u> between two sets $R_1$ and $R_2$ is a space $S$ and a pair of maps

$$
\begin{tikzcd}
 & S \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
R_1 & & R_2
\end{tikzcd}
$$

It generalizes the notion of a function from $R_1 \to R_2$. In the case of a function $S$ would be the graph and $f_1, f_2$ would be projection to domain and codomain.

To check things like gluing we would like to be able to compose correspondences:

Want to go from

$$\begin{tikzcd} & S_{12} \arrow[dl, "f_1"'] \arrow[dr, "f_{12}"] & & S_{23} \arrow[dl, "f_{23}"'] \arrow[dr, "f_3"] & \\ R_1 & & R_2 & & R_3 \end{tikzcd}$$

to

$$\begin{tikzcd} & & S_{13} \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & & \\ & S_{12} \arrow[dl, "f_1"'] \arrow[dr, "f_{12}"] & & S_{23} \arrow[dl, "f_{23}"'] \arrow[dr, "f_3"] & \\ R_1 & & R_2 & & R_3 \end{tikzcd}$$

In the world of topological spaces and continuous maps a natural way to do this is via the <u>homotopy fiber product</u>

$$\begin{tikzcd} & S_1 \times_h S_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S_1 \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\ & R & \end{tikzcd}$$

$$S_1 \times_h S_2 = \left\{ (s_1, s_2, \gamma) : \begin{array}{l} \gamma \text{ is a} \\ \text{continuous} \\ \text{path in } R: \end{array} \right\}$$
$$\gamma : f_1(s_1) \rightsquigarrow f_2(s_2)$$

Contrast this with the ordinary fiber product

$$S_1 \times_{f_1 f_2} S_2 = \{ (s_1, s_2) \mid f_1(s_1) = f_2(s_2) \}$$

# 9. Defects & Domain Walls

## In Finite Homotopy Theories

(Following Freed–Moore–Teleman, 2209.0747)

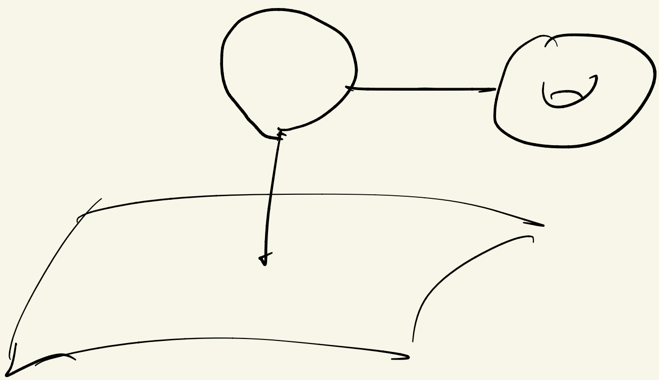
In the FHT $\sigma_{\mathcal{X}}^{(m)}$ we essentially introduce a notion of "dynamical fields" (up to homotopy)

$$\mathcal{X}^M = \mathrm{Map}(M, \mathcal{X})$$

We can do that with defects and domain walls as well. Then we "quantize" the relevant spaces and correspondences by taking functions, category of vector bundles, etc. on $\mathcal{X}^M$ – up to homotopy.

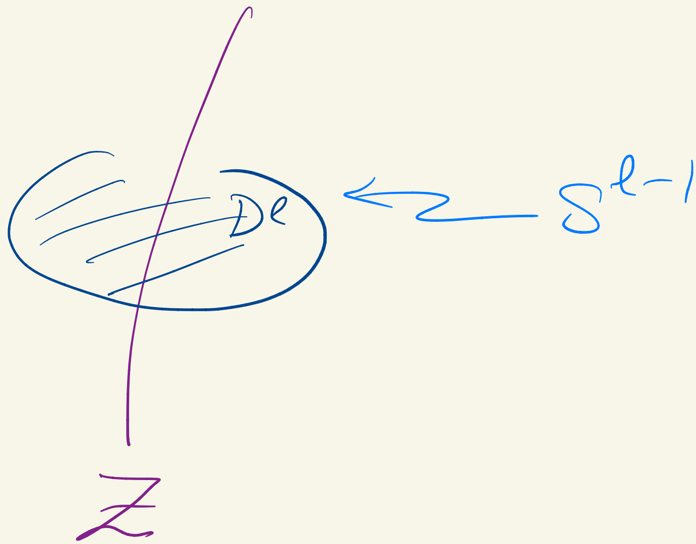
In general a defect will be associated to some subset $Z$ in a spacetime. $Z$ need not be smooth — it could be a stratified space

$Z$:



It is good to describe the defects first when $Z$ is a manifold and then piece them together working up in codimension.

If $Z$ is a smooth codim $= \ell$ submanifold of spacetime then locally it has a "linking sphere" $\cong S^{\ell-1}$




The "semi-classical" (dynamical) local degrees of freedom are declared to be a space $\mathcal{Y}$ and a map

$$\Psi : \mathcal{Y} \longrightarrow \mathcal{X}^{S^{\ell-1}}$$

To describe a defect globally

$$\mathcal{M} = \Big\{ (\phi_{bulk}, \phi_{defect}) \;\Big|$$

$$\phi_{bulk}: M_m \to \mathcal{X} \qquad \phi_{defect}: Z \to \mathcal{Y}$$

such that

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "\psi"] \\ Z \arrow[ur, "\phi_{defect}"] \arrow[r, "\phi_{bulk,Z}"'] & \mathcal{X}^{S^{l-1}} \end{tikzcd} \Big\}$$

Amplitudes, statespaces, etc. in the presence of the defect are obtained by "quantizing $\mathcal{M}$"

- Domain Walls
- Boundary Theories
- Dirichlet + Neumann Boundary Theories
- Example of $\sigma^{(m)}_{BG}$:

Reduction of structure group on the boundary

- Composition of defects
- Example of composition of domain walls between finite gauge theories

# 10. Symmetry Action Of A Finite Homotopy Theory On A QFT: The Quiche Picture

- Motivation 1: $G$-symmetry in QM

- Motivation 2: $SU(N)$ vs $PSU(N)$ gauge theory in 4d: Coupling to the 5d gerbe theory $\sigma^{(5)}_{B^2A}$ $\quad A \subset Z(SU(N)) \cong \mathbb{Z}_N$

- General definition of quiche and quiche action